

Contribution of Light-by-Light Scattering to Energy Levels of Light Muonic Atoms

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The complete contribution of diagrams with the light-by-light scattering to the Lamb shift is found for muonic hydrogen, deuterium and helium ion. The results are obtained in the static muon approximation and a part of the paper is devoted to the verification of this approximation and analysis of its uncertainty.

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Studies of energy levels in muonic hydrogen have a long history, but until recently there have been no successful precision measurements on this atom. Recently results on the Lamb shift in muonic hydrogen and deuterium were obtained for the first time [1], and similar measurements for muonic helium are planned.

The Lamb shift in light muonic two-body atoms is a splitting of the levels $2s$ and $2p$, which are degenerate within a nonrelativistic (NR) treatment of the Coulomb problem (as well as, e.g., in common hydrogen). Effects of the electron vacuum polarization in muonic atoms break down the degeneration and lead to the splitting of order $\alpha(Z\alpha)^2 m_\mu$, which can still be obtained from NR calculations. (The relativistic units, in which $\hbar = c = 1$, Z is the nuclear charge and α is the fine structure constant, are applied throughout the paper.)

There are also “finer” splittings and, in particular, levels in light muonic atoms possess the fine and hyperfine structure, which result from relativistic calculations and appear to be substantially smaller than the Lamb shift (in contrast to common hydrogen).

Specificity of light muonic atoms is that the characteristic atomic momentum is $Z\alpha m_\mu$, comparable to the mass of electron ($\alpha m_\mu \simeq 1.5 m_e$). At the same time, atomic energies are much lower than m_e .

Therefore, one can consider corrections to energy with closed electron loops as nonrelativistic. Effects of the vacuum polarization up to NR contributions to the Lamb shift of order $\alpha^5 m_\mu$ were studied in a number of papers (see, e.g., [2–5]).

While calculating $\alpha^5 m_\mu$ terms in the papers mentioned, the atomic nucleus was treated as a point-like one. Nuclear-finite-size effects in light muonic atoms can be considered, if necessary, as an additional perturbation.

Contributions, induced by the light-by-light (LbL) scattering (see Fig. 1), appear in order of $\alpha^5 m_\mu$ and are considered in this paper. The atomic nucleus in diagrams in Fig. 1 is treated in the external field approximation (i.e., as a static one), while the expression for the muonic line includes the Coulomb Green function of muon, which

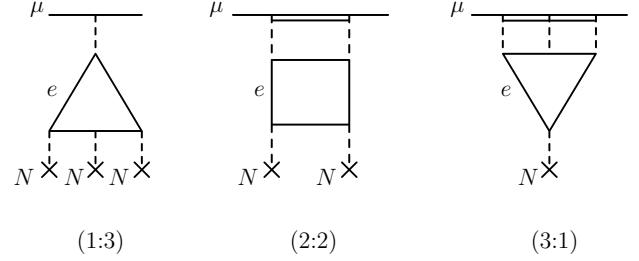


FIG. 1: Three types of diagrams for the LbL contributions to the Lamb shift in light muonic atoms

is indicated by the double line,

$$S_C(E, \mathbf{p}, \mathbf{p}') = i \sum_{\lambda} \frac{|\lambda(\mathbf{p})\rangle \langle \lambda(\mathbf{p}')|}{E - E_{\lambda} + i0}. \quad (1)$$

The sum is taken over all intermediate states λ of a discrete and continuous spectrum.

Each of the photons, including those that connect the muon line and the electron loop as well as those of the external nuclear field, are Coulomb photons, i.e., only the D_{00} component of the photon propagator contributes to the result. D_{00} does not depend on energy in the Coulomb gauge, however, energy is transferred through photons lines that connect the muon line and the electron loop. Apparently, energy does not propagate through the external field lines.

Generally speaking, since the muon atomic momentum is of order of the electron loop momenta, explicit forms of the functions $|\lambda(\mathbf{p})\rangle$ are substantially different for free and bound (Coulomb) wave functions (see, e.g., [6]).

Calculations with Coulomb Green functions turn out to be rather complicated even in the NR approximation; up to date they have not been performed for any of the corresponding contributions.

However, it is possible to demonstrate that the calculations can be made in a simple approximation considering muon as a static particle (see Fig. 2).

In this approximation the expressions for individual

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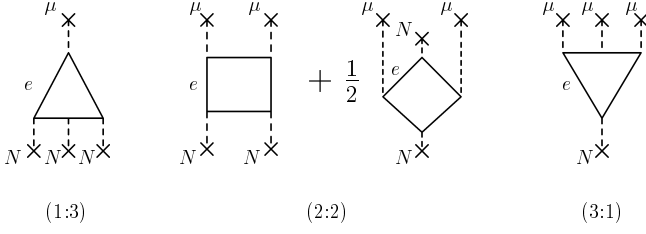


FIG. 2: Diagrams for the calculation of the Lamb shift contribution in the static muon approximation

contributions to the energy of the level ζ are of the form

$$\Delta E_{1:3}(\zeta) = (4\pi\alpha)(-4\pi Z\alpha)^3 \int \frac{d^3 k_1}{(2\pi)^3} \int \frac{d^3 k_2}{(2\pi)^3} \int \frac{d^3 k_3}{(2\pi)^3} \times \frac{1}{\mathbf{k}_1^2 \mathbf{k}_2^2 \mathbf{k}_3^2 \mathbf{k}_4^2} \mathcal{F}_\zeta(\mathbf{k}_1) \cdot \mathcal{L}, \quad (2)$$

$$\Delta E_{2:2}(\zeta) = (4\pi\alpha)^2 (-4\pi Z\alpha)^2 \int \frac{d^3 k_1}{(2\pi)^3} \int \frac{d^3 k_2}{(2\pi)^3} \int \frac{d^3 k_3}{(2\pi)^3} \times \frac{1}{\mathbf{k}_1^2 \mathbf{k}_2^2 \mathbf{k}_3^2 \mathbf{k}_4^2} \times \left[\mathcal{F}_\zeta(\mathbf{k}_1 + \mathbf{k}_2) + \frac{1}{2} \mathcal{F}_\zeta(\mathbf{k}_1 + \mathbf{k}_3) \right] \cdot \mathcal{L}, \quad (3)$$

$$\Delta E_{3:1}(\zeta) = (4\pi\alpha)^3 (-4\pi Z\alpha) \int \frac{d^3 k_1}{(2\pi)^3} \int \frac{d^3 k_2}{(2\pi)^3} \int \frac{d^3 k_3}{(2\pi)^3} \times \frac{1}{\mathbf{k}_1^2 \mathbf{k}_2^2 \mathbf{k}_3^2 \mathbf{k}_4^2} \mathcal{F}_\zeta(\mathbf{k}_4) \cdot \mathcal{L}, \quad (4)$$

where

$$\mathcal{F}_\zeta(\mathbf{q}) = \int \frac{d^3 p}{(2\pi)^3} \Psi_\zeta^*(\mathbf{p}) \Psi_\zeta(\mathbf{p} + \mathbf{q}) \quad (5)$$

is the atomic form factor, which is for the $2p-2s$ splitting found to be

$$\mathcal{F}_{2p-2s}(\mathbf{q}) = \frac{2\gamma^4 \mathbf{q}^2 (\gamma^2 - \mathbf{q}^2)}{(\gamma^2 + \mathbf{q}^2)^4}, \quad (6)$$

$\gamma = Z\alpha m_r$, and m_r is the reduced mass of the muon in the atom.

The factor $\mathcal{L}(k_1, k_2, k_3, k_4)$ corresponds to a single diagram of the LbL scattering with the momenta of the incoming photons defined as $k_i = (0, \mathbf{k}_i)$ for $i = 1, 2, 3$ and $k_4 = -(k_1 + k_2 + k_3)$ (see Fig. 3). The permutations of photons, as it is demonstrated below, have been already taken into account explicitly.

We remark that both the Coulomb Green functions and integrations over photon energy that are included in the full expressions have completely vanished. Note that energy in the static approximation does not propagate through photon lines, but does through photons that connect the muon and electron.

The static muon approximation qualitatively changes the form of expressions and notably simplifies the calculations.

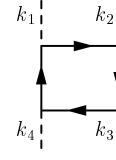


FIG. 3: The diagram for the LbL scattering block $\mathcal{L}(k_1, k_2, k_3, k_4)$

We present here the proof of its applicability to light muonic atoms, assuming at the first stage that the bound energy of the state ζ , for which the energy is calculated, as well as for intermediate states λ , is negligible. Verification of this assumption is discussed separately afterwards.

The standard muonic Coulomb Green function appears in our calculations with the following arguments:

$$S_C(E_\zeta \pm \omega, \mathbf{p}, \mathbf{p}'),$$

where ω is a photon frequency or combination of such values. In contrast to the static muon approximation (see Fig. 2), these frequencies are not zeros in the full expression.

Neglecting the energies of the states and taking into account completeness of the basis, one can find that the sum over λ becomes trivial:

$$\sum_\lambda \frac{|\lambda(\mathbf{p})\rangle \langle \lambda(\mathbf{p}')|}{E_\zeta \pm \omega - E_\lambda + i0} \rightarrow \sum_\lambda \frac{|\lambda(\mathbf{p})\rangle \langle \lambda(\mathbf{p}')|}{\pm \omega + i0} = \frac{(2\pi)^3 \delta(\mathbf{p} - \mathbf{p}')}{\pm \omega + i0}. \quad (7)$$

Expressions with the Coulomb Green function differ from expressions in the static muon approximation because three additional elements appear:

- 1) a sum over intermediate states;
- 2) an integration over momentum (note, while a muon propagates at the Coulomb field, its energy rather than momentum is conserved);
- 3) an integration over the photon frequencies (note that the virtual photons, connecting the muon and electron in Fig. 1, transfer energy, while the photons in the static approximation in Fig. 2 do not).

One should also take into account contributions of all possible permutations of the photon lines. We treat in our calculations a single diagram in Fig. 3 as the whole LbL scattering block, and all the permutations are taken into account as contributions to the muonic factor with permuted lines of outgoing photons.

The transformation (7) allows removing both the sum over intermediate states and “superfluous” integrations over momentum. Now the only difference between the expressions (3) and (4) and the Coulomb one is the integration over energy of the photons. (The expression (2) for the 1:3-contribution is already equal to the full expression since it does not really include the Coulomb Green function from the beginning.)

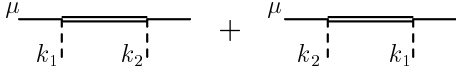


FIG. 4: The muon factor for the 2:2 contribution

Let us look at the structure of integrations over ω . Exchange of energy between the muon and electron is present in the Coulomb diagrams, but complete energy transfer from the nucleus to electron is equal to zero, and therefore the sum of energies of all photons connecting the electron and muon is zero as well. Thus, in the case of the 2:2 contribution, having defined the energy of the first photon k_{10} as ω (so that $k_{20} = -\omega$), we find that the sum of the direct and cross diagrams for the muon factor (see Fig. 4) leads to a δ function

$$\frac{i}{\omega + i0} + \frac{i}{-\omega + i0} = 2\pi\delta(\omega) \quad (8)$$

that removes the last “superfluous” integration. As a result, we obtain the expression (3).

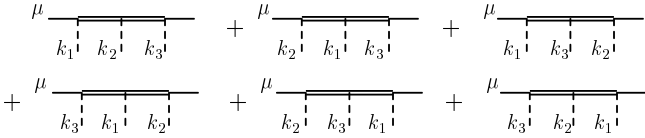


FIG. 5: The muon factor for the 3:1 contribution

The structure of the 3:1 diagrams is more complicated. Having defined energies of the photons as $k_{10} = \omega_1$, $k_{20} = \omega_2$ (so that $k_{30} = -(\omega_1 + \omega_2)$), the sum of six permutations for the total muon factor (see Fig. 5) takes the form

$$\begin{aligned} & \frac{i}{\omega_1 + i0} \frac{i}{\omega_1 + \omega_2 + i0} + \frac{i}{\omega_2 + i0} \frac{i}{\omega_1 + \omega_2 + i0} \\ & + \frac{i}{\omega_1 + i0} \frac{i}{-\omega_2 + i0} + \frac{i}{-\omega_1 - \omega_2 + i0} \frac{i}{-\omega_2 + i0} \\ & + \frac{i}{\omega_2 + i0} \frac{i}{-\omega_1 + i0} + \frac{i}{-\omega_1 - \omega_2 + i0} \frac{i}{-\omega_1 + i0} \\ & = \left(\frac{i}{\omega_1 - i0} - \frac{i}{\omega_1 + i0} \right) \left(\frac{i}{\omega_2 - i0} - \frac{i}{\omega_2 + i0} \right) \\ & = 2\pi\delta(\omega_1) \times 2\pi\delta(\omega_2). \end{aligned} \quad (9)$$

The δ functions remove integrations over energy and lead to the static expression (4).

Therefore, neglecting the energy of the states in the Coulomb Green function (7), we have in fact turned the Coulomb diagrams in Fig. 1 into diagrams in Fig. 2 with a static muon. We still have to verify that it is really possible to neglect the dismissed terms and estimate the error of this procedure.

In case of the 2:2-contribution the structure of the integral over the photon energy for individual diagrams is

of the form

$$\mathcal{I}(\epsilon) = \int d\omega \frac{1}{\pm\omega + \epsilon + i0} \mathcal{L}(\omega), \quad (10)$$

where we define the combination of energies that we are to neglect as $\epsilon \ll m_e$.

The first factor is for the muon and it is distinct for different contributions and the second one, $\mathcal{L}(\omega)$, which corresponds to the LbL scattering block, is universal.

After combining with other diagrams the first factor leads to the δ function at zero frequency (cf. (8) and (9)), therefore, strictly speaking, it cannot be expanded. One can see that after performing a substitution of the variable by shifting $\omega \rightarrow \omega' \mp \epsilon$, the integral turns to

$$\mathcal{I}(\epsilon) = \int d\omega' \frac{1}{\pm\omega' + i0} \mathcal{L}(\omega' \pm \epsilon). \quad (11)$$

This expression can already be expanded in powers of ϵ .

It is clear that while integrating in the LbL block, the characteristic scale of loop energies in $\mathcal{L}(\omega)$ is of order of the electron mass (or higher) and the parameter of expansion is

$$\frac{\epsilon}{m_e} \sim \frac{(Z\alpha)^2 m_\mu}{m_e} \simeq 0.01 \cdot Z^2.$$

It is clear that a similar approach with substitution of a variable is applicable for the 3:1 correction as well. E.g., for the first term in (9) one can write

$$\begin{aligned} & \int d\omega_1 \int d\omega_2 \frac{1}{\omega_1 + \epsilon_1 + i0} \frac{\mathcal{L}(\omega_1, \omega_2)}{\omega_1 + \omega_2 + \epsilon_2 + i0} \\ & = \int d\omega'_1 \int d\omega'_2 \frac{1}{\omega'_1 + i0} \frac{\mathcal{L}(\omega'_1 - \epsilon_1, \omega'_2 - \epsilon_2 + \epsilon_1)}{\omega'_1 + \omega'_2 + i0}, \end{aligned}$$

and then the expansion in ϵ can be done without problems.

We note that, if the diagrams in Fig. 1 contain the free muon propagators instead of the Coulomb Green functions, the muon static approximation is applicable and the uncertainty is of the same order. Dealing with the the muon Green function (see, e.g., (7)), only the completeness of the eigenstate basis is required, while the estimated eigenvalues of the energy are of the same order for the free and Coulomb case, as long as the related integrals are convergent.

One of direct consequences of the applicability of the static muon approximation and of symmetry of the expressions (2) and (4) is the identity [5]

$$\Delta E_{3:1} = \frac{1}{Z^2} \Delta E_{1:3}. \quad (12)$$

Until recently the 3:1 term has been the only completely unknown contribution. On the contrary, the 1:3-contribution, which is also referred to as the Wichmann-Kroll contribution, has been well known (see [6–8] for results in muonic hydrogen, [9] in muonic deuterium and

[5, 10] muonic helium). That is due to the fact that approximations of the Wichmann-Kroll potential in the form of an explicit function of distance, which are efficient for calculations, have been known [11, 12]. There is also an exact representation of this potential in the form of a double integral [13], understood in terms of the principal value.

The identity (12) changes the situation radically. It by itself improves accuracy of contributions to the Lamb shift in muonic hydrogen and muonic deuterium by an order of magnitude and also leads to a certain improvement of the accuracy for muonic helium.

According to Eq. (12), the 2:2 term, which is also referred to as the virtual Delbrück scattering, becomes the least accurately known contribution from the LbL block. The results for muonic hydrogen [14] and deuterium [9] are obtained with accuracy less than 10%, and for muonic helium-4 ion it has been rather an estimate [10] than a result obtained.

The 2:2 contribution was examined in the cited papers in the so called scattering approximation, where at first an operator related to the Feynman diagram with external muon lines on the mass shell and with a free muon propagator, is derived, and then its matrix element over the Coulomb wave function is calculated. In fact, before the calculation was really done [12, 16], a few additional approximations had been introduced, which reduced the applied expressions to the static muon approximation.

Leaving aside the question about reasonability of using the scattering approximation as an initial point, we state that direct calculations in [9, 10, 14] were in fact performed in the static muon approximation, which is correct for the corresponding Coulomb diagrams in Fig. 1 within the declared accuracy of calculations.

The 2:2 contribution is one of the most complex specific QED contributions in light muonic atoms and absence of independent confirmations significantly reduces reliability of the results obtained in [9, 10, 14].

Below we calculate the 1:3 and 2:2 terms. The first calculation is used for control of analytical expressions and numerical algorithms. The second one is aimed to improve accuracy of the total contribution of the diagram in Fig. 1 for light muonic atoms, since the accuracy of known results is not high enough.

To perform a calculation, one has at first to find an efficient form of the factor \mathcal{L} , which corresponds to the LbL scattering diagram (see Fig. 3). A calculation within the standard technique of Feynman parameterization leads to the result

$$\begin{aligned} \mathcal{L} &= \mathcal{L}(m_e) - \mathcal{L}(M), \\ \mathcal{L}(m) &= \frac{1}{8\pi^2} \int_0^1 dx \int_0^1 dy \int_0^1 dz x^2 y \\ &\times \left[\frac{N_3}{2} \frac{m^4}{\Delta^2} - N_4 \left(\frac{1}{\Delta} + \frac{m^2}{\Delta^2} \right) + \frac{N_6}{2} \frac{1}{\Delta^2} \right] \end{aligned} \quad (13)$$

where

$$\begin{aligned} \Delta &= m^2 - \mathbf{Q}^2 + x (\mathbf{k}_1^2 + 2y(\mathbf{k}_1\mathbf{k}_2) \\ &\quad + y\mathbf{k}_2^2 + yz\mathbf{k}_3^2 + 2yz\mathbf{k}_3(\mathbf{k}_1 + \mathbf{k}_2)) , \\ \mathbf{Q} &= x (\mathbf{k}_1 + y\mathbf{k}_2 + yz\mathbf{k}_3) , \\ N_3 &= 4 , \\ N_4 &= -2 \left[(\mathbf{Q} - \mathbf{k}_2) (\mathbf{Q} - \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \right. \\ &\quad \left. + \mathbf{Q} (\mathbf{Q} - \mathbf{k}_1) - \mathbf{k}_1 (\mathbf{Q} - \mathbf{k}_1 - \mathbf{k}_2) \right] , \\ N_6 &= 4 \left[(\mathbf{Q} \cdot (\mathbf{Q} - \mathbf{k}_1)) ((\mathbf{Q} - \mathbf{k}_1 - \mathbf{k}_2) \right. \\ &\quad \cdot (\mathbf{Q} - \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)) \\ &\quad - (\mathbf{Q} \cdot (\mathbf{Q} - \mathbf{k}_1 - \mathbf{k}_2)) \\ &\quad \times ((\mathbf{Q} - \mathbf{k}_1) \cdot (\mathbf{Q} - \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)) \\ &\quad \left. + (\mathbf{Q} \cdot (\mathbf{Q} - \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)) \right. \\ &\quad \left. \times ((\mathbf{Q} - \mathbf{k}_1) \cdot (\mathbf{Q} - \mathbf{k}_1 - \mathbf{k}_2)) \right] . \end{aligned}$$

Here, we have explicitly introduced the Pauli-Villars regularization with $M \gg m_e$, which can be useful if the convergence of the integrals over k is not good enough.

It is useful to make a further calculation by combining the photon denominators and the denominator Δ using Feynman parameters. After integration over momenta we come to the following expression:

$$\Delta E_{1:3}(\zeta) = \frac{3}{4\pi} \alpha (Z\alpha)^3 \int \frac{d^3 q}{(2\pi)^3 \mathbf{q}^2} \mathcal{F}_\zeta(\mathbf{q}) \mathcal{J}_{1:3} , \quad (14)$$

where

$$\begin{aligned} \mathcal{J}_{1:3}(\mathbf{q}^2) &= \int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 du \int_0^1 dv \int_0^1 dw \\ &\times \left\{ \mathcal{A}_{1:3} \left[\ln \left(\frac{s_{1:3} \mathbf{q}^2 + m_e^2}{m_e^2} \right) - \ln \left(\frac{M^2}{m_e^2} \right) \right] \right. \\ &\quad \left. + \frac{\mathcal{B}_{1:3} \mathbf{q}^2}{(s_{1:3} \mathbf{q}^2 + m_e^2)} + \frac{\mathcal{C}_{1:3} \mathbf{q}^4}{(s_{1:3} \mathbf{q}^2 + m_e^2)^2} \right\} \end{aligned} \quad (15)$$

and the dimensionless coefficients $\mathcal{A}_{1:3}$, $\mathcal{B}_{1:3}$, $\mathcal{C}_{1:3}$ and $s_{1:3}$ are bulky functions of all Feynman parameters.

Calculating the 1:3 contribution, the integration over the momentum of the atomic form factor \mathcal{F} in (2) is factorized, and the remaining integrations over momenta involve logarithmic divergencies at large momenta. Therefore, keeping M at intermediate stages turns to be useful for a calculation of individual terms. In particular, contributions to (14) of the separate terms of (13) include such divergencies.

The value $\mathcal{J}_{1:3}(\mathbf{q}^2)$ is nothing else but a contribution to the charge form factor of the muon induced by the LbL

block. We can always “renormalize” the vertex function of muon and, by subtracting

$$\mathcal{J}_{1:3}(\mathbf{q}^2) \rightarrow \mathcal{J}_{1:3}(\mathbf{q}^2) - \mathcal{J}_{1:3}(0), \quad (16)$$

remove the logarithmic divergency in integration over k in (15).

In fact, the LbL scattering diagram, as it is known, does not renormalize the vertex, i.e., $\mathcal{J}_{1:3}(0) = 0$, that we have checked both numerically and analytically at different stages of transformations. In particular, the coefficient at the logarithm $\ln(M/m_e)$ in (15) turns to zero

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 du \int_0^1 dv \int_0^1 dw \mathcal{A}_{1:3} = 0.$$

Therefore “renormalization” of individual terms according to (16) is actually reduced to regrouping and cancellation of individual divergent terms.

The final integration over Feynman parameters and momentum q has been performed by numerical integration with VEGAS [15]. The results are presented in Table I. The contributions for the $2p - 2s$ splitting in the last line of the table were calculated directly rather than as the difference of the $2p$ and $2s$ terms.

Level(s)	$C(\mu\text{H})$	$C(\mu\text{D})$	$C(\mu\text{He})$
$2s$	0.5704(5)	0.6263(5)	1.0815(10)
$2p$	0.10468(3)	0.12405(3)	0.5108(7)
$2p - 2s$	-0.4649(5)	-0.5015(5)	-0.5702(10)

TABLE I: The 1:3 contribution (the Wichmann-Kroll contribution) to energy levels of muonic hydrogen, deuterium and helium-4 ion: $\Delta E = \alpha(Z\alpha)^4 m_\mu \cdot 10^{-3} \cdot C$

The results for the Lamb shift $2p - 2s$ (see also Table III) are in excellent agreement with the values previously obtained by other authors for muonic hydrogen ($-0.00103(2)$ meV [6, 8]) and deuterium ($-0.00111(2)$ meV [9]). For helium ion our result agrees with $-0.0198(4)$ meV [5] and -0.02 meV [10], but disagrees with the result 0.135 meV [17]. The uncertainty of the results [8, 9] was not present in the original papers and was estimated here according to [5].

The momentum integrations for the 2:2 contribution were performed similarly to those described above. One can check that it is possible here to set $M = \infty$ before integration over large momenta. After introducing Feynman parameters and integrating over momenta, we obtain

$$\Delta E_{2:2}(\zeta) = \frac{3}{4\pi} \alpha(Z\alpha)^3 \int \frac{d^3 q}{(2\pi)^3} \mathcal{F}_\zeta(\mathbf{q}) \mathcal{J}_{2:2},$$

where

$$\begin{aligned} \mathcal{J}_{2:2}(\mathbf{q}^2) = & \int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 du \int_0^1 dv \int_0^1 dw \int_0^1 dt \\ & \times \sum_{k=1,2} \left\{ \frac{\mathcal{B}_{2:2}^{(k)}}{\left(s_{2:2}^{(k)} \mathbf{q}^2 + m_e^2\right)} + \frac{\mathcal{C}_{2:2}^{(k)} \mathbf{q}^2}{\left(s_{2:2}^{(k)} \mathbf{q}^2 + m_e^2\right)^2} \right. \\ & \left. + \frac{\mathcal{D}_{2:2}^{(k)} \mathbf{q}^4}{\left(s_{2:2}^{(k)} \mathbf{q}^2 + m_e^2\right)^3} \right\}. \end{aligned} \quad (17)$$

The coefficients $\mathcal{B}_{2:2}^{(k)}$, $\mathcal{C}_{2:2}^{(k)}$, $\mathcal{D}_{2:2}^{(k)}$ and $s_{2:2}^{(k)}$ are bulky dimensionless functions of the Feynman parameters. Note that the denominators for the ladder ($k = 1$) and non-ladder ($k = 2$) diagrams are slightly different.

The integration over q and the Feynman parameters is performed numerically. The results are presented in Table II. Similarly to the case of the 1:3 contribution, the value for the $2p - 2s$ splitting in the table is found by a direct numerical integration.

Level(s)	$C(\mu\text{H})$	$C(\mu\text{D})$	$C(\mu\text{He})$
$2s$	-0.8201(15)	-0.9000(15)	-1.5615(25)
$2p$	-0.2937(9)	-0.3355(9)	-0.908(2)
$2p - 2s$	0.5264(12)	0.5645(12)	0.652(2)

TABLE II: The 2:2 contribution (the virtual Delbrück scattering) to energy levels of muonic hydrogen, deuterium and helium-4 ion in the static muon approximation: $\Delta E = \alpha^2(Z\alpha)^3 m_\mu \cdot 10^{-3} \cdot C$

The results for $2p - 2s$ in the static muon approximation are slightly smaller and have substantially higher accuracy than the results of the other authors. Ours are mostly in agreement with the former values. In particular, for muonic hydrogen we obtain $0.001151(4)$ meV (cf. $0.00135(15)$ meV [14]), for deuterium, $0.001234(4)$ meV (cf. $0.00147(16)$ meV [9]) and for helium ion, $0.01140(4)$ meV (cf. 0.02 meV [10]).

Complete contributions of the diagram in Fig. 1 to the Lamb shift are collected in Table III. In contrast to Table II, where the uncertainty of numerical calculations in the static muon approximation is presented, here we take into account the error of the static muon approximation, which dominates.

In conclusion, we note smallness of the numerical coefficients in Tables I and II, which is typical for spin-independent nonrelativistic contributions of the light-by-light scattering to energy levels.

As we have noted, the results for the 2:2 contribution obtained in this paper are slightly below the results of Borie [9, 10, 14]. The difference for muonic hydrogen and deuterium is about 20%, i.e. less than 1.5 standard deviations. In the case of muonic helium our result ($0.01140(2)$ meV) is also lower than that of Borie (0.02 meV), however, the latter is presented in [10] in

Term	$\Delta E(\mu\text{H})$ [meV]	$\Delta E(\mu\text{D})$ [meV]	$\Delta E(\mu\text{He})$ [meV]
1:3	-0.001 018(4)	-0.001 098(4)	-0.019 95(6)
2:2	0.001 15(1)	0.001 24(1)	0.0114(4)
3:1	-0.001 02(1)	-0.001 10(1)	-0.0050(2)
Total	-0.000 89(2)	-0.000 96(2)	-0.0136(6)

TABLE III: Contribution of the LbL scattering effects (Fig. 1) to the Lamb shift ($2p - 2s$) in muonic hydrogen, deuterium and helium-4 ion

such a form that one is rather able to estimate the scale of uncertainty than its magnitude.

This difference in the results can in no way be considered as a contradiction, but it has a systematic character (all our results are lower than those of Borie) and, strictly speaking, requires some explanation.

Due to that, we have checked two most important elements of our calculations, namely, the expression for the

LbL scattering (13) and the integration over momenta k_i in (4). As a test of the representation for \mathcal{L} we have used a calculation of the well-known 1:3 contribution (see Table I) and studied the expression (13) for the LbL scattering in the case when momentum of one of the photons is zero.

To be certain that there are no systematic errors in integration over momenta after introducing Feynman parameters we have made an independent calculation of the 2:2 contribution, in which integrations over momenta k_i in the corresponding expression (4) were performed directly. The results turn out to have accuracy 4% for muonic hydrogen and 3% for muonic helium ion and are in excellent agreement with results of Table II.

These tests allow to consider our results to be highly reliable.

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